CHAPTER 9

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

INTRODUCTORY REMARK

Thus far we have concentrated on first-order differential equations. We will now turn our attention to the second-order case. After investigating solution techniques, we will discuss applications of these differential equations (see Chapter 14).

THE CHARACTERISTIC EQUATION

Corresponding to the differential equation

$$y'' + a_1y' + a_0y = 0 \tag{9.1}$$

in which a_1 and a_0 are constants, is the algebraic equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{9.2}$$

which is obtained from Eq. (*9.1*) by replacing y'', y' and y by λ^2, λ^1, and $\lambda^0 = 1$, respectively. Equation (*9.2*) is called the *characteristic equation* of (*9.1*).

Example 9.1. The characteristic equation of $y'' + 3y' - 4y = 0$ is $\lambda^2 + 3\lambda - 4 = 0$; the characteristic equation of $y'' - 2y' + y = 0$ is $\lambda^2 - 2\lambda + 1 = 0$.

Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j ($j = 0, 1, 2$).

The characteristic equation can be factored into

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \tag{9.3}$$

THE GENERAL SOLUTION

The general solution of (*9.1*) is obtained directly from the roots of (*9.3*). There are three cases to consider.

Case 1. λ_1 and λ_2 both real and distinct. Two linearly independent solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, and the general solution is (Theorem 8.2)

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{9.4}$$

In the special case $\lambda_2 = -\lambda_1$, the solution (*9.4*) can be rewritten as $y = k_1 \cosh \lambda_1 x + k_2 \sinh \lambda_1 x$.

Case 2. $\lambda_1 = a + ib$, a complex number. Since a_1 and a_0 in (*9.1*) and (*9.2*) are assumed real, the roots of (*9.2*) must appear in conjugate pairs; thus, the other root is $\lambda_2 = a - ib$. Two linearly independent solutions are $e^{(a+ib)x}$ and $e^{(a-ib)x}$, and the general complex solution is

$$y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x} \tag{9.5}$$

which is algebraically equivalent to (see Problem 9.16)

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \tag{9.6}$$

Case 3. $\lambda_1 = \lambda_2$. Two linearly independent solutions are $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$, and the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \tag{9.7}$$

Warning: The above solutions *are not valid* if the differential equation is not linear or does not have constant coefficients. Consider, for example, the equation $y'' - x^2 y = 0$. The roots of the characteristic equation are $\lambda_1 = x$ and $\lambda_2 = -x$, but the solution is *not*

$$y = c_1 e^{(x)x} + c_2 e^{(-x)x} = c_1 e^{x^2} + c_2 e^{-x^2}$$

Linear equations with variable coefficients are considered in Chapters 27, 28 and 29.

Solved Problems

9.1. Solve $y'' - y' - 2y = 0$.

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, which can be factored into $(\lambda + 1)(\lambda - 2) = 0$. Since the roots $\lambda_1 = -1$ and $\lambda_2 = 2$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{-x} + c_2 e^{2x}$$

9.2. Solve $y'' - 7y' = 0$.

The characteristic equation is $\lambda^2 - 7\lambda = 0$, which can be factored into $(\lambda - 0)(\lambda - 7) = 0$. Since the roots $\lambda_1 = 0$ and $\lambda_2 = 7$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{0x} + c_2 e^{7x} = c_1 + c_2 e^{7x}$$

9.3. Solve $y'' - 5y = 0$.

The characteristic equation is $\lambda^2 - 5 = 0$, which can be factored into $(\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$. Since the roots $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}$$

9.4. Rewrite the solution of Problem 9.3 in terms of hyperbolic functions.

Using the results of Problem 9.3 with the identities

$$e^{\lambda x} = \cosh \lambda x + \sinh \lambda x \quad \text{and} \quad e^{-\lambda x} = \cosh \lambda x - \sinh \lambda x$$

we obtain,

$$\begin{aligned} y &= c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x} \\ &= c_1(\cosh \sqrt{5}x + \sinh \sqrt{5}x) + c_2(\cosh \sqrt{5}x - \sinh \sqrt{5}x) \\ &= (c_1 + c_2)\cosh \sqrt{5}x + (c_1 - c_2)\sinh \sqrt{5}x \\ &= k_1 \cosh \sqrt{5}x + k_2 \sinh \sqrt{5}x \end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = c_1 - c_2$.

9.5. Solve $\ddot{y} + 10\dot{y} + 21y = 0$.

Here the independent variable is t. The characteristic equation is

$$\lambda^2 + 10\lambda + 21 = 0$$

which can be factored into

$$(\lambda + 3)(\lambda + 7) = 0$$

The roots $\lambda_1 = -3$ and $\lambda_2 = -7$ are real and distinct, so the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-7t}$$

9.6. Solve $\ddot{x} - 0.01x = 0$.

The characteristic equation is

$$\lambda^2 - 0.01 = 0$$

which can be factored into

$$(\lambda - 0.1)(\lambda + 0.1) = 0$$

The roots $\lambda_1 = 0.1$ and $\lambda_2 = -0.1$ are real and distinct, so the general solution is

$$y = c_1 e^{0.1t} + c_2 e^{-0.1t}$$

or, equivalently,

$$y = k_1 \cosh 0.1t + k_2 \sinh 0.1t$$

9.7. Solve $y'' + 4y' + 5y = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(4) \pm \sqrt{(4)^2 - 4(5)}}{2} = -2 \pm i$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) (with $a = -2$ and $b = 1$) as

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

9.8. Solve $y'' + 4y = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda = 0$$

which can be factored into

$$(\lambda - 2i)(\lambda + 2i) = 0$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) (with $a = 0$ and $b = 2$) as

$$y = c_1 \cos 2x + c_2 \sin 2x$$

9.9. Solve $y'' - 3y' + 4y = 0$.

The characteristic equation is

$$\lambda^2 - 3\lambda + 4 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) as

$$y = c_1 e^{(3/2)x} \cos \frac{\sqrt{7}}{2} x + c_2 e^{(3/2)x} \sin \frac{\sqrt{7}}{2} x$$

9.10. Solve $\ddot{y} - 6\dot{y} + 25y = 0$.

The characteristic equation is

$$\lambda^2 - 6\lambda + 25 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(25)}}{2} = 3 \pm i4$$

These roots are a complex conjugate pair, so the general solution is

$$y = c_1 e^{3t} \cos 4t + c_2 e^{3t} \sin 4t$$

9.11. Solve $\frac{d^2 I}{dt^2} + 20\frac{dI}{dt} + 200I = 0$.

The characteristic equation is

$$\lambda^2 - 20\lambda + 200 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(20) \pm \sqrt{(20)^2 - 4(200)}}{2} = -10 \pm i10$$

These roots are a complex conjugate pair, so the general solution is

$$I = c_1 e^{-10t} \cos 10t + c_2 e^{-10t} \sin 10t$$

9.12. Solve $y'' - 8y' + 16y = 0$.

The characteristic equation is

$$\lambda^2 - 8\lambda + 16 = 0$$

which can be factored into

$$(\lambda - 4)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = 4$ are real and equal, so the general solution is given by (*9.7*) as

$$y = c_1e^{4x} + c_2xe^{4x}$$

9.13. Solve $y'' = 0$.

The characteristic equation is $\lambda^2 = 0$, which has roots $\lambda_1 = \lambda_2 = 0$. The solution is given by (*9.7*) as

$$y = c_1e^{0x} + c_2xe^{0x} = c_1 + c_2x$$

9.14. Solve $\ddot{x} + 4\dot{x} + 4x = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

which can be factored into

$$(\lambda + 2)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = -2$ are real and equal, so the general solution is

$$x = c_1e^{-2t} + c_2te^{-2t}$$

9.15. Solve $100\dfrac{d^2N}{dt^2} - 20\dfrac{dN}{dt} + N = 0$.

Dividing both sides of the differential equation by 100, to force the coefficient of the highest derivative to be unity, we obtain

$$\frac{d^2N}{dt^2} - 0.2\frac{dN}{dt} + 0.01N = 0$$

Its characteristic equation is

$$\lambda^2 - 0.2\lambda + 0.01 = 0$$

which can be factored into

$$(\lambda - 0.1)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = 0.1$ are real and equal, so the general solution is

$$N = c_1e^{-0.1t} + c_2te^{-0.1t}$$

9.16. Prove that (*9.6*) is algebraically equivalent to (*9.5*).

Using Euler's relations

$$e^{ibx} = \cos bx + i \sin bx \quad e^{-ibx} = \cos bx - i \sin bx$$

we can rewrite (*9.5*) as

$$\begin{aligned} y &= d_1e^{ax}e^{ibx} + d_2e^{ax}e^{-ibx} = e^{ax}(d_1e^{ibx} + d_2e^{-ibx}) \\ &= e^{ax}[d_1(\cos bx + i\sin bx) + d_2(\cos bx - i\sin bx)] \\ &= e^{ax}[(d_1 + d_2)\cos bx + i(d_1 - d_2)\sin bx] \\ &= c_1e^{ax}\cos bx + c_2e^{ax}\sin bx \qquad (1) \end{aligned}$$

where $c_1 = d_1 + d_2$ and $c_2 = i(d_1 - d_2)$.

Equation (*1*) is real if and only if c_1 and c_2 are both real, which occurs, if and only if d_1 and d_2 are complex conjugates. Since we are interested in the general *real* solution to (*9.1*), we restrict d_1 and d_2 to be a conjugate pair.

Supplementary Problems

Solve the following differential equations.

9.17. $y'' - y = 0$

9.18. $y'' - y' - 30y = 0$

9.19. $y'' - 2y' + y = 0$

9.20. $y'' + y = 0$

9.21. $y'' + 2y' + 2y = 0$

9.22. $y'' - 7y = 0$

9.23. $y'' + 6y' + 9y = 0$

9.24. $y'' + 2y' + 3y = 0$

9.25. $y'' - 3y' - 5y = 0$

9.26. $y'' + y' + \frac{1}{4}y = 0$

9.27. $\ddot{x} - 20\dot{x} + 64x = 0$

9.28. $\ddot{x} + 60\dot{x} + 500x = 0$

9.29. $\ddot{x} - 3\dot{x} + x = 0$

9.30. $\ddot{x} - 10\dot{x} + 25x = 0$

9.31. $\ddot{x} + 25x = 0$

9.32. $\ddot{x} + 25\dot{x} = 0$

9.33. $\ddot{x} + \dot{x} + 2x = 0$

9.34. $\ddot{u} - 2\dot{u} + 4u = 0$

9.35. $\ddot{u} - 4\dot{u} + 2u = 0$

9.36. $\ddot{u} - 36\dot{u} = 0$

9.37. $\ddot{u} - 36u = 0$

9.38. $\frac{d^2Q}{dt^2} - 5\frac{dQ}{dt} + 7Q = 0$

9.39. $\frac{d^2Q}{dt^2} - 7\frac{dQ}{dt} + 5Q = 0$

9.40. $\frac{d^2P}{dt^2} - 18\frac{dP}{dt} + 81P = 0$

9.41. $\frac{d^2P}{dx^2} + 2\frac{dP}{dx} + 9P = 0$

9.42. $\frac{d^2N}{dx^2} + 5\frac{dN}{dx} - 24N = 0$

9.43. $\frac{d^2N}{dx^2} + 5\frac{dN}{dx} + 24N = 0$

9.44. $\frac{d^2T}{d\theta^2} + 30\frac{dT}{d\theta} + 225T = 0$

9.45. $\frac{d^2R}{d\theta^2} + 5\frac{dR}{d\theta} = 0$

CHAPTER 10

*n*th-Order Linear Homogeneous Differential Equations with Constant Coefficients

THE CHARACTERISTIC EQUATION

The characteristic equation of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \tag{10.1}$$

with constant coefficients a_j $(j = 0, 1, \ldots, n-1)$ is

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{10.2}$$

The characteristic equation *(10.2)* is obtained from *(10.1)* by replacing $y^{(j)}$ by λ^j $(j = 0, 1, \ldots, n-1)$. Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j $(j = 0, 1, \ldots, n-1)$.

Example 10.1. The characteristic equation of $y^{(4)} - 3y''' + 2y'' - y = 0$ is $\lambda^4 - 3\lambda^3 + 2\lambda^2 - 1 = 0$. The characteristic equation of

$$\frac{d^5x}{dt^5} - 3\frac{d^3x}{dt^3} + 5\frac{dx}{dt} - 7x = 0$$

is

$$\lambda^5 - 3\lambda^3 + 5\lambda - 7 = 0$$

Caution: Characteristic equations are only defined for linear homogeneous differential equations with constant coefficients.

THE GENERAL SOLUTION

The roots of the characteristic equation determine the solution of (*10.1*). If the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and distinct, the solution is

$$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + \cdots + c_ne^{\lambda_n x} \tag{10.3}$$

If the roots are distinct, but some are complex, then the solution is again given by (*10.3*). As in Chapter 9, those terms involving complex exponentials can be combined to yield terms involving sines and cosines. If λ_k is a root of multiplicity p [that is, if $(\lambda - \lambda_k)^p$ is a factor of the characteristic equation, but $(\lambda - \lambda_k)^{p+1}$ is not] then there will be p linearly independent solutions associated with λ_k given by $e^{\lambda_k x}, xe^{\lambda_k x}, x^2e^{\lambda_k x}, \ldots, x^{p-1}e^{\lambda_k x}$. These solutions are combined in the usual way with the solutions associated with the other roots to obtain the complete solution.

In theory it is always possible to factor the characteristic equation, but in practice this can be extremely difficult, especially for differential equations of high order. In such cases, one must often use numerical techniques to approximate the solutions. See Chapters 18, 19 and 20.

Solved Problems

10.1. Solve $y''' - 6y'' + 11y' - 6y = 0$.

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, which can be factored into

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The roots are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$; hence the solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

10.2. Solve $y^{(4)} - 9y'' + 20y = 0$.

The characteristic equation is $\lambda^4 - 9\lambda^2 + 20 = 0$, which can be factored into

$$(\lambda - 2)(\lambda + 2)(\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$$

The roots are $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = \sqrt{5}$, and $\lambda_4 = -\sqrt{5}$; hence the solution is

$$\begin{aligned} y &= c_1e^{2x} + c_2e^{-2x} + c_3e^{\sqrt{5}x} + c_4e^{-\sqrt{5}x} \\ &= k_1\cosh 2x + k_2\sinh 2x + k_3\cosh\sqrt{5}x + k_4\sinh\sqrt{5}x \end{aligned}$$

10.3. Solve $y' - 5y = 0$.

The characteristic equation is $\lambda - 5 = 0$, which has the single root $\lambda_1 = 5$. The solution is then $y = c_1e^{5x}$. (Compare this result with Problem 6.9.)

10.4. Solve $y''' - 6y'' + 2y' + 36y = 0$.

The characteristic equation, $\lambda^3 - 6\lambda^2 + 2\lambda + 36 = 0$, has roots $\lambda_1 = -2$, $\lambda_2 = 4 + i\sqrt{2}$, and $\lambda_3 = 4 - i\sqrt{2}$. The solution is

$$y = c_1e^{-2x} + d_2e^{(4+i\sqrt{2})x} + d_3e^{(4-i\sqrt{2})x}$$

which can be rewritten, using Euler's relations (see Problem 9.16) as

$$y = c_1e^{-2x} + c_2e^{4x}\cos\sqrt{2}x + c_3e^{4x}\sin\sqrt{2}x$$

10.5. Solve $\dfrac{d^4x}{dt^4} - 4\dfrac{d^3x}{dt^3} + 7\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 6x = 0.$

The characteristic equation, $\lambda^4 - 4\lambda^3 + 7\lambda^2 - 4\lambda + 6 = 0$, has roots $\lambda_1 = 2 + i\sqrt{2}$, $\lambda_2 = 2 - i\sqrt{2}$, $\lambda_3 = i$, and $\lambda_4 = -i$. The solution is

$$x = d_1e^{(2+i\sqrt{2})t} + d_2e^{(2-i\sqrt{2})t} + d_3e^{it} + d_4e^{-it}$$

If, using Euler's relations, we combine the first two terms and then similarly combine the last two terms, we can rewrite the solution as

$$x = c_1e^{2t}\cos\sqrt{2}t + c_2e^{2x}\sin\sqrt{2}t + c_3\cos t + c_4\sin t$$

10.6. Solve $y^{(4)} + 8y''' + 24y'' + 32y' + 16y = 0.$

The characteristic equation, $\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda + 16 = 0$, can be factored into $(\lambda + 2)^4 = 0$. Here $\lambda_1 = -2$ is a root of multiplicity four; hence the solution is

$$y = c_1e^{-2x} + c_2xe^{-2x} + c_3x^2e^{-2x} + c_4x^3e^{-2x}$$

10.7. Solve $\dfrac{d^5P}{dt^5} - \dfrac{d^4P}{dt^4} - 2\dfrac{d^3P}{dt^3} + 2\dfrac{d^2P}{dt^2} + \dfrac{dP}{dt} - P = 0.$

The characteristic equation can be factored into $(\lambda - 1)^3(\lambda + 1)^2 = 0$; hence, $\lambda_1 = 1$ is a root of multiplicity three and $\lambda_2 = -1$ is a root of multiplicity two. The solution is

$$P = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{-t} + c_5te^{-t}$$

10.8. Solve $\dfrac{d^4Q}{dx^4} - 8\dfrac{d^3Q}{dx^3} + 32\dfrac{d^2Q}{dx^2} - 64\dfrac{dQ}{dx} + 64Q = 0.$

The characteristic equation has roots $2 \pm i2$ and $2 \pm i2$; hence $\lambda_1 = 2 + i2$ and $\lambda_2 = 2 - i2$ are both roots of multiplicity two. The solution is

$$\begin{aligned} Q &= d_1e^{(2+i2)x} + d_2xe^{(2+i2)x} + d_3e^{(2-i2)x} + d_4xe^{(2-i2)x} \\ &= e^{2x}(d_1e^{i2x} + d_3e^{-i2x}) + xe^{2x}(d_2e^{i2x} + d_4e^{-i2x}) \\ &= e^{2x}(c_1\cos 2x + c_3\sin 2x) + xe^{2x}(c_2\cos 2x + c_4\sin 2x) \\ &= (c_1 + c_2x)e^{2x}\cos 2x + (c_3 + c_4x)e^{2x}\sin 2x \end{aligned}$$

10.9. **Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be x^3e^{4x}.**

If x^3e^{4x} is a solution, then so too are x^2e^{4x}, xe^{4x}, and e^{4x}. We now have four linearly independent solutions to a fourth-order linear, homogeneous differential equation, so we can write the general solution as

$$y(x) = c_4x^3e^{4x} + c_3x^2e^{4x} + c_2xe^{4x} + c_1e^{4x}$$

10.10. **Determine the differential equation described in Problem 10.9.**

The characteristic equation of a fourth-order differential equation is a fourth-degree polynomial having exactly four roots. Because x^3e^{4x} is a solution, we know that $\lambda = 4$ is a root of multiplicity four of the corresponding characteristic equation, so the characteristic equation must be $(\lambda - 4)^4 = 0$, or

$$\lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = 0$$

The associated differential equation is

$$y^{(4)} - 16y''' + 96y'' - 256y' + 256y = 0$$

10.11. Find the general solution to a third-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are known to be e^{-2x} and $\sin 3x$.

If $\sin 3x$ is a solution, then so too is $\cos 3x$. Together with e^{-2x}, we have three linearly independent solutions to a third-order linear, homogeneous differential equation, and we can write the general solution as

$$y(x) = c_1 e^{-2x} + c_2 \cos 3x + c_3 \sin 3x$$

10.12. Determine the differential equation described in Problem 10.11.

The characteristic equation of a third-order differential equation must have three roots. Because e^{-2x} and $\sin 3x$ are solutions, we know that $\lambda = -2$ and $\lambda = \pm i3$ are roots of the corresponding characteristic equation, so this equation must be

$$(\lambda + 2)(\lambda - i3)(\lambda + i3) = 0$$

or

$$\lambda^3 + 2\lambda^2 + 9\lambda + 18 = 0$$

The associated differential equation is

$$y''' + 2y'' + 9y' + 18y = 0$$

10.13. Find the general solution to a sixth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be $x^2e^{7x} \cos 5x$.

If $x^2e^{7x} \cos 5x$ is a solution, then so too are $xe^{7x} \cos 5x$ and $e^{7x} \cos 5x$. Furthermore, because complex roots of a characteristic equation come in conjugate pairs, every solution containing a cosine term is matched with another solution containing a sine term. Consequently, $x^2e^{7x} \sin 5x$, $xe^{7x} \sin 5x$, and $e^{7x} \sin 5x$ are also solutions. We now have six linearly independent solutions to a sixth-order linear, homogeneous differential equation, so we can write the general solution as

$$y(x) = c_1x^2e^{7x} \cos 5x + c_2x^2e^{7x} \sin 5x + c_3xe^{7x} \cos 5x + c_4xe^{7x} \sin 5x + c_5e^{7x} \cos 5x + c_6e^{7x} \sin 5x$$

10.14. Redo Problem 10.13 if the differential equation has order 8.

An eighth-order linear differential equation possesses eight linearly independent solutions, and since we can only identify six of them, as we did in Problem 10.13, we do not have enough information to solve the problem. We can say that the solution to Problem 10.13 will be *part* of the solution to this problem.

10.15. Solve $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 36\dfrac{dy}{dx} - 36y = 0$ if one solution is xe^{2x}.

If xe^{2x} is a solution, then so too is e^{2x} which implies that $(\lambda - 2)^2$ is a factor of the characteristic equation $\lambda^4 - 4\lambda^3 - 5\lambda^2 + 36\lambda - 36 = 0$. Now,

$$\frac{\lambda^4 - 4\lambda^3 - 5\lambda^2 + 36\lambda - 36}{(\lambda - 2)^2} = \lambda^2 - 9$$

so two other roots of the characteristic equation are $\lambda = \pm 3$, with corresponding solutions e^{3x} and e^{-3x}. Having identified four linearly independent solutions to the given fourth-order linear differential equation, we can write the general solution as

$$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{3x} + c_4e^{-3x}$$

Supplementary Problems

In Problems 10.16 through 10.34, solve the given differential equations.

10.16. $y''' - 2y'' - y' + 2y = 0$

10.17. $y''' - y'' - y' + y = 0$

10.18. $y''' - 3y'' + 3y' - y = 0$

10.19. $y''' - y'' + y' - y = 0$

10.20. $y^{(4)} + 2y'' + y = 0$

10.21. $y^{(4)} - y = 0$

10.22. $y^{(4)} + 2y''' - 2y' - y = 0$

10.23. $y^{(4)} - 4y'' + 16y' + 32y = 0$

10.24. $y^{(4)} + 5y''' = 0$

10.25. $y^{(4)} + 2y''' + 3y'' + 2y' + y = 0$

10.26. $y^{(6)} - 5y^{(4)} + 16y''' + 36y'' - 16y' - 32y = 0$

10.27. $\dfrac{d^4x}{dt^4} + 4\dfrac{d^3x}{dt^3} + 6\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + x = 0$

10.28. $\dfrac{d^3x}{dt^3} = 0$

10.29. $\dfrac{d^4x}{dt^4} + 10\dfrac{d^2x}{dt^2} + 9x = 0$

10.30. $\dfrac{d^3x}{dt^3} - 5\dfrac{d^2x}{dt^2} + 25\dfrac{dx}{dt} - 125x = 0$

10.31. $q^{(4)} + q'' - 2q = 0$

10.32. $q^{(4)} - 3q'' + 2q = 0$

10.33. $N''' - 12N'' - 28N + 480N = 0$

10.34. $\dfrac{d^5r}{d\theta^5} + 5\dfrac{d^4r}{d\theta^4} + 10\dfrac{d^3r}{d\theta^3} + 10\dfrac{d^2r}{d\theta^2} + 5\dfrac{dr}{d\theta} + r = 0$

In Problems 10.35 through 10.41, a complete set of roots is given for the characteristic equation of an nth-order near homogeneous differential equation in $y(x)$ with real numbers as coefficients. Determine the general solution of the differential equation.

10.35. $2, 8, -14$

10.36. $0, \pm i19$

10.37. $0, 0, 2 \pm i9$

10.38. $2 \pm i9, 2 \pm i9$

10.39. $5, 5, 5, -5, -5$

10.40. $\pm i6, \pm i6, \pm i6$

10.41. $-3 \pm i, -3 \pm i, 3 \pm i, 3 \pm i$

10.42. Determine the differential equation associated with the roots given in Problem 10.35.

10.43. Determine the differential equation associated with the roots given in Problem 10.36.

10.44. Determine the differential equation associated with the roots given in Problem 10.37.

10.45. Determine the differential equation associated with the roots given in Problem 10.38.

10.46. Determine the differential equation associated with the roots given in Problem 10.39.

10.47. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be x^3e^{-x}.

10.48. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are $\cos 4x$ and $\sin 3x$.

10.49. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is $x \cos 4x$.

10.50. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are xe^{2x} and xe^{5x}.

CHAPTER 11

The Method of Undetermined Coefficients

The general solution to the linear differential equation $\mathbf{L}(y) = \phi(x)$ is given by Theorem 8.4 as $y = y_h + y_p$ where y_p denotes one solution to the differential equation and y_h is the general solution to the associated homogeneous equation, $\mathbf{L}(y) = 0$. Methods for obtaining y_h when the differential equation has constant coefficients are given in Chapters 9 and 10. In this chapter and the next, we give methods for obtaining a particular solution y_p *once* y_h *is known.*

SIMPLE FORM OF THE METHOD

The *method of undetermined coefficients* is applicable only if $\phi(x)$ and *all* of its derivatives can be written in terms of the same *finite* set of linearly independent functions, which we denote by $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. The method is initiated by assuming a particular solution of the form

$$y_p(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x)$$

where $A_1, A_2, \ldots, A_n$ denote arbitrary multiplicative constants. These arbitrary constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

Case 1. $\boldsymbol{\phi(x) = p_n(x)}$**, an** $\boldsymbol{n}$**th-degree polynomial in** $\boldsymbol{x}$**.** Assume a solution of the form

$$y_p = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 \tag{11.1}$$

where A_j ($j = 0, 1, 2, \ldots, n$) is a constant to be determined.

Case 2. $\boldsymbol{\phi(x) = ke^{\alpha x}}$ **where** $\boldsymbol{k}$ **and** $\boldsymbol{\alpha}$ **are known constants.** Assume a solution of the form

$$y_p = Ae^{\alpha x} \tag{11.2}$$

where A is a constant to be determined.

Case 3. $\boldsymbol{\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x}$ **where** $\boldsymbol{k_1, k_2}$**, and** $\boldsymbol{\beta}$ **are known constants.** Assume a solution of the form

$$y_p = A \sin \beta x + B \cos \beta x \tag{11.3}$$

where A and B are constants to be determined.

Note: (*11.3*) in its entirety is assumed even when k_1 or k_2 is zero, because the derivatives of sines or cosines involve both sines and cosines.

GENERALIZATIONS

If $\phi(x)$ is the product of terms considered in Cases 1 through 3, take y_p to be the product of the corresponding assumed solutions and algebraically combine arbitrary constants where possible. In particular, if $\phi(x) = e^{\alpha x}p_n(x)$ is the product of a polynomial with an exponential, assume

$$y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \tag{11.4}$$

where A_j is as in Case 1. If, instead, $\phi(x) = e^{\alpha x}p_n(x) \sin \beta x$ is the product of a polynomial, exponential, and sine term, or if $\phi(x) = e^{\alpha x}p_n(x) \cos \beta x$ is the product of a polynomial, exponential, and cosine term, then assume

$$y_p = e^{\alpha x} \sin \beta x\, (A_n x^n + \cdots + A_1 x + A_0) + e^{\alpha x} \cos \beta x\, (B_n x^n + \cdots + B_1 x + B_0) \tag{11.5}$$

where A_j and B_j ($j = 0, 1, \ldots, n$) are constants which still must be determined.

If $\phi(x)$ is the sum (or difference) of terms already considered, then we take y_p to be the sum (or difference) of the corresponding assumed solutions and algebraically combine arbitrary constants where possible.

MODIFICATIONS

If any term of the assumed solution, disregarding multiplicative constants, is also a term of y_h (the homogeneous solution), then the assumed solution must be modified by multiplying it by x^m, where m is the smallest positive integer such that the product of x^m with the assumed solution has no terms in common with y_h.

LIMITATIONS OF THE METHOD

In general, if $\phi(x)$ is not one of the types of functions considered above, or if the differential equation *does not have constant coefficients*, then the method given in Chapter 12 applies.

Solved Problems

11.1. Solve $y'' - y' - 2y = 4x^2$.

From Problem 9.1, $y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x) = 4x^2$, a second-degree polynomial. Using (*11.1*), we assume that

$$y_p = A_2 x^2 + A_1 x + A_0 \tag{1}$$

Thus, $y_p' = 2A_2 x + A_1$ and $y_p'' = 2A_2$. Substituting these results into the differential equation, we have

$$2A_2 - (2A_2 x + A_1) - 2(A_2 x^2 + A_1 x + A_0) = 4x^2$$

or, equivalently,

$$(-2A_2)x^2 + (-2A_2 - 2A_1)x + (2A_2 - A_1 - 2A_0) = 4x^2 + (0)x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2A_2 = 4 \qquad -2A_2 - 2A_1 = 0 \qquad 2A_2 - A_1 - 2A_0 = 0$$

Solving this system, we find that $A_2 = -2$, $A_1 = 2$, and $A_0 = -3$. Hence (*1*) becomes

$$y_p = -2x^2 + 2x - 3$$

and the general solution is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

11.2. Solve $y'' - y' - 2y = e^{3x}$.

From Problem 9.1, $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x)$ has the form displayed in Case 2 with $k = 1$ and $\alpha = 3$. Using (*11.2*), we assume that

$$y_p = Ae^{3x} \tag{1}$$

Thus, $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting these results into the differential equation, we have

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \quad \text{or} \quad 4Ae^{3x} = e^{3x}$$

It follows that $4A = 1$, or $A = \frac{1}{4}$, so that (*1*) becomes $y_p = \frac{1}{4}e^{3x}$. The general solution then is

$$y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$$

11.3. Solve $y'' - y' - 2y = \sin 2x$.

Again by Problem 9.1, $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x)$ has the form displayed in Case 3 with $k_1 = 1$, $k_2 = 0$, and $\beta = 2$. Using (*11.3*), we assume that

$$y_p = A\sin 2x + B\cos 2x \tag{1}$$

Thus, $y_p' = 2A\cos 2x - 2B\sin 2x$ and $y_p'' = -4A\sin 2x - 4B\cos 2x$. Substituting these results into the differential equation, we have

$$(-4A\sin 2x - 4B\cos 2x) - (2A\cos 2x - 2B\sin 2x) - 2(A\sin 2x + B\cos 2x) = \sin 2x$$

or, equivalently,

$$(-6A + 2B)\sin 2x + (-6B - 2A)\cos 2x = (1)\sin 2x + (0)\cos 2x$$

Equating coefficients of like terms, we obtain

$$-6A + 2B = 1 \qquad -2A - 6B = 0$$

Solving this system, we find that $A = -3/20$ and $B = 1/20$. Then from (*1*),

$$y_p = -\frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

and the general solution is

$$y = c_1e^{-x} + c_2e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

11.4. Solve $\ddot{y} - 6\dot{y} + 25y = 2\sin\frac{t}{2} - \cos\frac{t}{2}$.

From Problem 9.10,

$$y_h = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$$

Here $\phi(t)$ has the form displayed in Case 3 with the independent variable t replacing x, $k_1 = 2$, $k_2 = -1$, and $\beta = \frac{1}{2}$. Using (*11.3*), with t replacing x, we assume that

$$y_p = A\sin\frac{t}{2} + B\cos\frac{t}{2} \tag{1}$$

Consequently,

$$\dot{y}_p = \frac{A}{2}\cos\frac{t}{2} - \frac{B}{2}\sin\frac{t}{2}$$

and

$$\ddot{y}_p = -\frac{A}{4}\sin\frac{t}{2} - \frac{B}{4}\cos\frac{t}{2}$$

Substituting these results into the differential equation, we obtain

$$\left(-\frac{A}{4}\sin\frac{t}{2} - \frac{B}{4}\cos\frac{t}{4}\right) - 6\left(\frac{A}{2}\cos\frac{t}{2} - \frac{B}{2}\sin\frac{t}{2}\right) + 25\left(A\sin\frac{t}{2} + B\cos\frac{t}{2}\right) = 2\sin\frac{t}{2} - \cos\frac{t}{2}$$

or, equivalently

$$\left(\frac{99}{4}A + 3B\right)\sin\frac{t}{2} + \left(-3A + \frac{99}{4}B\right)\cos\frac{t}{2} = 2\sin\frac{t}{2} - \cos\frac{t}{2}$$

Equating coefficients of like terms, we have

$$\frac{99}{4}A + 3B = 2; \quad -3A + \frac{99}{4}B = -1$$

It follows that $A = 56/663$ and $B = -20/663$, so that (*1*) becomes

$$y_p = \frac{56}{663}\sin\frac{t}{2} - \frac{20}{663}\cos\frac{t}{2}$$

The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + \frac{56}{663}\sin\frac{t}{2} - \frac{20}{663}\cos\frac{t}{2}$$

11.5. Solve $\ddot{y} - 6\dot{y} + 25y = 64e^{-t}$.

From Problem 9.10,

$$y_h = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$$

Here $\phi(t)$ has the form displayed in Case 2 with the independent variable t replacing x, $k = 64$ and $\alpha = -1$. Using (*11.2*), with t replacing x, we assume that

$$y_p = Ae^{-t} \tag{1}$$

Consequently, $\dot{y}_p = -Ae^{-t}$ and $\ddot{y}_p = Ae^{-t}$. Substituting these results into the differential equation, we obtain

$$Ae^{-t} - 6(-Ae^{-t}) + 25(Ae^{-t}) = 64e^{-t}$$

or, equivalently, $32Ae^{-t} = 64e^{-t}$. It follows that $32A = 64$ or $A = 2$, so that (*1*) becomes $y_p = 2e^{-t}$. The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + 2e^{-t}$$

11.6. Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18$.

Again by Problem 9.10,

$$y_h = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$$

Here $\phi(t)$ is a third-degree polynomial in t. Using (*11.1*) with t replacing x, we assume that

$$y_p = A_3t^3 + A_2t^2 + A_1t + A_0 \tag{1}$$

Consequently,

$$\dot{y}_p = 3A_3t^2 + 2A_2t + A_1$$

and

$$\ddot{y}_p = 6A_3t + 2A_2$$

Substituting these results into the differential equation, we obtain

$$(6A_3t + 2A_2) - 6(3A_3t^2 + 2A_2t + A_1) + 25(A_3t^3 + A_2t^2 + A_1t + A_0) = 50t^3 - 36t^2 - 63t + 18$$

or, equivalently,

$$(25A_3)t^3 + (-18A_3 + 25A_2)t^2 + (6A_3 - 12A_2 + 25A_1) + (2A_2 - 6A_1 + 25A_0) = 50t^3 - 36t^2 - 63t + 18$$

Equating coefficients of like powers of t, we have

$$25A_3 = 50; \quad -18A_3 + 25A_2 = -36; \quad 6A_3 - 12A_2 + 25A_1 = -63; \quad 2A_2 - 6A_1 + 25A_0 = 18$$

Solving these four algebraic equations simultaneously, we obtain $A_3 = 2$, $A_2 = 0$, $A_1 = -3$, and $A_0 = 0$, so that (*1*) becomes

$$y_p = 2t^3 - 3t$$

The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + 2t^3 - 3t$$

11.7. Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

From Problem 10.1, $y_h = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Here $\phi(x) = e^{\alpha x}p_n(x)$, where $\alpha = -1$ and $p_n(x) = 2x$, a first-degree polynomial. Using Eq. (*11.4*), we assume that $y_p = e^{-x}(A_1x + A_0)$, or

$$y_p = A_1xe^{-x} + A_0e^{-x} \tag{1}$$

Thus,

$$y_p' = -A_1xe^{-x} + A_1e^{-x} - A_0e^{-x}$$

$$y_p'' = A_1xe^{-x} - 2A_1e^{-x} + A_0e^{-x}$$

$$y_p''' = -A_1xe^{-x} + 3A_1e^{-x} - A_0e^{-x}$$

Substituting these results into the differential equation and simplifying, we obtain

$$-24A_1xe^{-x} + (26A_1 - 24A_0)e^{-x} = 2xe^{-x} + (0)e^{-x}$$

Equating coefficients of like terms, we have

$$-24A_1 = 2 \quad 26A_1 - 24A_0 = 0$$

from which $A_1 = -1/12$ and $A_0 = -13/144$.

Equation (*1*) becomes

$$y_p = -\frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

and the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

11.8. Determine the form of a particular solution for $y'' = 9x^2 + 2x - 1$.

Here $\phi(x) = 9x^2 + 2x - 1$, and the solution of the associated homogeneous differential equation $y'' = 0$ is $y_h = c_1x + c_0$. Since $\phi(x)$ is a second-degree polynomial, we first try $y_p = A_2x^2 + A_1x + A_0$. Note, however, that this assumed solution has terms, disregarding multiplicative constants, in common with y_h: in particular, the first-power term and the constant term. Hence, we must determine the smallest positive integer m such that $x^m(A_2x^2 + A_1x + A_0)$ has no terms in common with y_h.

For $m = 1$, we obtain

$$x(A_2x^2 + A_1x + A_0) = A_2x^3 + A_1x^2 + A_0x$$

which still has a first-power term in common with y_h. For $m = 2$, we obtain

$$x^2(A_2x^2 + A_1x + A_0) = A_2x^4 + A_1x^3 + A_0x^2$$

which has no terms in common with y_h; therefore, we assume an expression of this form for y_p.

11.9. Solve $y'' = 9x^2 + 2x - 1$.

Using the results of Problem 11.8, we have $y_h = c_1x + c_0$ and we assume

$$y_p = A_2x^4 + A_1x^3 + A_0x^2 \tag{1}$$

Substituting (*1*) into the differential equation, we obtain

$$12A_2x^2 + 6A_1x + 2A_0 = 9x^2 + 2x - 1$$

from which $A_2 = 3/4$, $A_1 = 1/3$, and $A_0 = -1/2$. Then (*1*) becomes

$$y_p = \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2$$

and the general solution is

$$y = c_1x + c_0 + \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2$$

The solution also can be obtained simply by twice integrating both sides of the differential equation with respect to x.

11.10. Solve $y' - 5y = 2e^{5x}$.

From Problem 10.3, $y_h = c_1e^{5x}$. Since $\phi(x) = 2e^{5x}$, it would follow from Eq. (*11.2*) that the guess for y_p should be $y_p = A_0e^{5x}$. Note, however, that this y_p has exactly the same form as y_h; therefore, we must modify y_p. Multiplying y_p by x ($m = 1$), we obtain

$$y_p = A_0xe^{5x} \tag{1}$$

As this expression has no terms in common with y_h; it is a candidate for the particular solution. Substituting (*1*) and $y_p' = A_0e^{5x} + 5A_0xe^{5x}$ into the differential equation and simplifying, we obtain $A_0e^{5x} = 2e^{5x}$, from which $A_0 = 2$. Equation (*1*) becomes $y_p = 2xe^{5x}$, and the general solution is $y = (c_1 + 2x)e^{5x}$.

11.11. Determine the form of a particular solution of

$$y' - 5y = (x - 1)\sin x + (x + 1)\cos x$$

Here $\phi(x) = (x - 1)\sin x + (x + 1)\cos x$, and from Problem 10.3, we know that the solution to the associated homogeneous problem $y' - 5y = 0$ is $y_h = c_1e^{5x}$. An assumed solution for $(x - 1)\sin x$ is given by Eq. (*11.5*) (with $\alpha = 0$) as

$$(A_1x + A_0)\sin x + (B_1x + B_0)\cos x$$

and an assumed solution for $(x + 1)\cos x$ is given also by Eq. (*11.5*) as

$$(C_1x + C_0)\sin x + (D_1x + D_0)\cos x$$

(Note that we have used C and D in the last expression, since the constants A and B already have been used.) We therefore take

$$y_p = (A_1x + A_0)\sin x + (B_1x + B_0)\cos x + (C_1x + C_0)\sin x + (D_1x + D_0)\cos x$$

Combining like terms, we arrive at

$$y_p = (E_1x + E_0)\sin x + (F_1x + F_0)\cos x$$

as the assumed solution, where $E_j = A_j + C_j$ and $F_j = B_j + D_j$ ($j = 0, 1$).

11.12. Solve $y' - 5y = (x - 1)\sin x + (x + 1)\cos x$.

From Problem 10.3, $y_h = c_1e^{5x}$. Using the results of Problem 11.11, we assume that

$$y_p = (E_1x + E_0)\sin x + (F_1x + F_0)\cos x \tag{1}$$

Thus, $$y_p' = (E_1 - F_1x - F_0)\sin x + (E_1x + E_0 + E_1)\cos x$$

Substituting these values into the differential equation and simplifying, we obtain

$$(-5E_1 - F_1)x\sin x + (-5E_0 + E_1 - F_0)\sin x + (-5F_1 + E_1)\,x\cos x + (-5F_0 + E_0 + F_1)\cos x$$
$$= (1)x\sin x + (-1)\sin x + (1)x\cos x + (1)\cos x$$

Equating coefficients of like terms, we have

$$\begin{aligned} -5E_1 - F_1 &= 1 \\ -5E_0 + E_1 - F_0 &= -1 \\ E_1 - 5F_1 &= 1 \\ E_0 - 5F_0 + F_1 &= 1 \end{aligned}$$

Solving, we obtain $E_1 = -2/13$, $E_0 = 71/338$, $F_1 = -3/13$, and $F_0 = -69/338$. Then, from (*1*),

$$y_p = \left(-\frac{2}{13}x + \frac{71}{338}\right)\sin x + \left(-\frac{3}{13}x + \frac{69}{338}\right)\cos x$$

and the general solution is

$$y = c_1e^{5x} + \left(\frac{-2}{13}x + \frac{71}{338}\right)\sin x - \left(\frac{3}{13}x + \frac{69}{338}\right)\cos x$$

11.13. Solve $y' - 5y = 3e^x - 2x + 1$.

From Problem 10.3, $y_h = c_1e^{5x}$. Here, we can write $\phi(x)$ as the sum of two manageable functions: $\phi(x) = (3e^x) + (-2x + 1)$. For the term $3e^x$ we would assume a solution of the form Ae^x; for the term $-2x + 1$ we would assume a solution of the form $B_1x + B_0$. Thus, we try

$$y_p = Ae^x + B_1x + B_0 \tag{1}$$

Substituting (*1*) into the differential equation and simplifying, we obtain

$$(-4A)e^x + (-5B_1)x + (B_1 - 5B_0) = (3)e^x + (-2)x + (1)$$

Equating coefficients of like terms, we find that $A = -3/4$, $B_1 = 2/5$, and $B_0 = -3/25$. Hence, (*1*) becomes

$$y_p = -\frac{3}{4}e^x + \frac{2}{5}x - \frac{3}{25}$$

and the general solution is

$$y = c_1 e^{5x} - \frac{3}{4}e^x + \frac{2}{5}x - \frac{3}{25}$$

11.14. Solve $y' - 5y = x^2 e^x - xe^{5x}$.

From Problem 10.3, $y_h = c_1 e^{5x}$. Here $\phi(x) = x^2 e^x - xe^{5x}$, which is the difference of two terms, each in manageable form. For $x^2 e^x$ we would assume a solution of the form

$$e^x(A_2x^2 + A_1x + A_0) \tag{1}$$

For xe^{5x} we would try initially a solution of the form

$$e^{5x}(B_1x+B_0) = B_1xe^{5x} + B_0e^{5x}$$

But this supposed solution would have, disregarding multiplicative constants, the term e^{5x} in common with y_h. We are led, therefore, to the modified expression

$$xe^{5x}(B_1x + B_0) = e^{5x}(B_1x^2 + B_0x) \tag{2}$$

We now take y_p to be the sum of (*1*) and (*2*):

$$y_p = e^x(A_2x^2 + A_1x + A_0) + e^{5x}(B_1x^2 + B_0x) \tag{3}$$

Substituting (*3*) into the differential equation and simplifying, we obtain

$$e^x[(-4A_2)x^2 + (2A_2 - 4A_1)x + (A_1 - 4A_0)] + e^{5x}[(2B_1)x + B_0]$$
$$= e^x[(1)x^2 + (0)x + (0)] + e^{5x}[(-1)x + (0)]$$

Equating coefficients of like terms, we have

$$-4A_2 = 1 \quad 2A_2 - 4A_1 = 0 \quad A_1 - 4A_0 = 0 \quad 2B_1 = -1 \quad B_0 = 0$$

from which

$$A_2 = -\frac{1}{4} \quad A_1 = -\frac{1}{8} \quad A_0 = -\frac{1}{32}$$
$$B_1 = -\frac{1}{2} \quad B_0 = 0$$

Equation (*3*) then gives

$$y_p = e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$$

and the general solution is

$$y = c_1e^{5x} + e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$$

Supplementary Problems

In Problems 11.15 through 11.26, determine the form of a particular solution to $\mathbf{L}(y) = \phi(x)$ for $\phi(x)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(y) = 0$ is $y_h = c_1e^{2x} + c_2e^{3x}$.

11.15. $\phi(x) = 2x - 7$

11.16. $\phi(x) = -3x^2$

11.17. $\phi(x) = 132x^2 - 388x + 1077$

11.18. $\phi(x) = 0.5e^{-2x}$

11.19. $\phi(x) = 13e^{5x}$

11.20. $\phi(x) = 4e^{2x}$

11.21. $\phi(x) = 2\cos 3x$

11.22. $\phi(x) = \frac{1}{2}\cos 3x - 3\sin 3x$

11.23. $\phi(x) = x\cos 3x$

11.24. $\phi(x) = 2x + 3e^{8x}$

11.25. $\phi(x) = 2xe^{5x}$

11.26. $\phi(x) = 2xe^{3x}$

In Problems 11.27 through 11.36, determine the form of a particular solution to $\mathbf{L}(y) = \phi(x)$ for $\phi(x)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(y) = 0$ is $y_h = c_1e^{5x}\cos 3x + c_2e^{5x}\sin 3x$.

11.27. $\phi(x) = 2e^{3x}$

11.28. $\phi(x) = xe^{3x}$

11.29. $\phi(x) = -23e^{5x}$

11.30. $\phi(x) = (x^2 - 7)e^{5x}$

11.31. $\phi(x) = 5\cos\sqrt{2}x$

11.32. $\phi(x) = x^2\sin\sqrt{2}x$

11.33. $\phi(x) = -\cos 3x$

11.34. $\phi(x) = 2\sin 4x - \cos 7x$

11.35. $\phi(x) = 31e^{-x}\cos 3x$

11.36. $\phi(x) = -\frac{1}{6}e^{5x}\cos 3x$

In Problems 11.37 through 11.43, determine the form of a particular solution to $\mathbf{L}(x) = \phi(t)$ for $\phi(t)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(x) = 0$ is $x_h = c_1 + c_2e^t + c_3te^t$.

11.37. $\phi(t) = t$

11.38. $\phi(t) = 2t^2 - 3t + 82$

11.39. $\phi(t) = te^{-2t} + 3$

11.40. $\phi(t) = -6e^t$

11.41. $\phi(t) = te^t$

11.42. $\phi(t) = 3 + t\cos t$

11.43. $\phi(t) = te^{2t}\cos 3t$

In Problems 11.44 through 11.52, find the general solutions to the given differential equations.

11.44. $y'' - 2y' + y = x^2 - 1$

11.45. $y'' - 2y' + y = 3e^{2x}$

11.46. $y'' - 2y' + y = 4\cos x$

11.47. $y'' - 2y' + y = 3e^x$

11.48. $y'' - 2y' + y = xe^x$

11.49. $y' - y = e^x$

11.50. $y' - y = xe^{2x} + 1$

11.51. $y' - y = \sin x + \cos 2x$

11.52. $y''' - 3y'' + 3y' - y = e^x + 1$